REIDEMEISTER TORSION AND DEHN SURGERY ON TWIST KNOTS

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Abstract. We compute the Reidemeister torsion of the complement of a twist knot in S^3 and that of the 3-manifold obtained by a Dehn surgery on a twist knot.

1. Main results

In a recent paper Kitano [Ki1] gives a formula for the Reidemeister torsion of the 3-manifold obtained by a Dehn surgery on the figure eight knot. In this paper we generalize his result to all twist knots. Specifically, we will compute the Reidemeister torsion of the complement of a twist knot in S^3 and that of the 3-manifold obtained by a Dehn surgery on a twist knot.

Let $J(k,l)$ be the link in Figure 1, where k,l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k,l)$ is a knot if and only if kl is even. The knot $J(2,2n)$, where $n \neq 0$, is known as a twist knot. For more information on $J(k,l)$, see [HS].

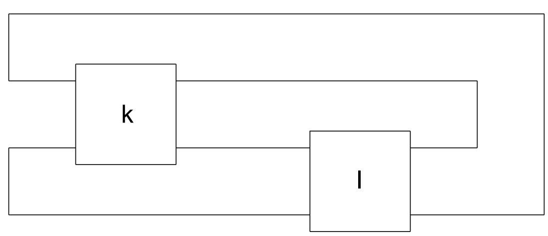

Figure 1. The link $J(k,l)$.

In this paper we fix $K = J(2,2n)$. Let E_K be the complement of K in S^3. The fundamental group of E_K has a presentation $\pi_1(E_K) = \langle a, b \mid w^n a = b w^n \rangle$ where a, b are meridians and $w = ba^{-1}b^{-1}a$. A representation $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is called nonabelian if the image of ρ is a nonabelian subgroup of $SL_2(\mathbb{C})$. Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ -u & s^{-1} \end{bmatrix}$$

where $(s,u) \in (\mathbb{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s,u)$, see [Ri].

Let $x := \operatorname{tr}\rho(a) = s + s^{-1}$ and $z := \operatorname{tr}\rho(w) = u^2 - (x^2-4)u + 2$. Let $S_k(z)$ be the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k.

2010 *Mathematics Classification:* Primary 57N10. Secondary 57M25.

Key words and phrases: Dehn surgery, nonabelian representation, Reidemeister torsion, twist knot.

Theorem 1. *Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is a nonabelian representation. If $x \neq 2$ then the Reidemeister torsion of E_K is given by*

$$\tau_\rho(E_K) = (2-x)\frac{S_n(z) - S_{n-2}(z) - 2}{z-2} + xS_{n-1}(z).$$

Now let M be the 3-manifold obtained by a $\frac{p}{q}$-surgery on the twist knot K. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle a, b \mid w^n a = bw^n, a^p\lambda^q = 1\rangle,$$

where λ is the canonical longitude corresponding to the meridian $\mu = a$.

Theorem 2. *Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. If $x \notin \{0, 2\}$ then the Reidemeister torsion of M is given by*

$$\tau_\rho(M) = \left((x-2)\frac{S_n(z) - S_{n-2}(z) - 2}{z-2} - xS_{n-1}(z)\right)\left(u^{-2}(u+1)(x^2-4) - 1\right)x^{-2}.$$

Remark 1.1. (1) One can see that the expression $(S_n(z) - S_{n-2}(z) - 2)/(z-2)$ is actually a polynomial in z.

(2) Theorem 2 generalizes the formula for the Reidemeister torsion of the 3-manifold obtained by a $\frac{p}{q}$-surgery on the figure eight knot by Kitano [Ki1].

Example 1.2. (1) If $n = 1$, then $K = J(2, 2)$ is the trefoil knot. In this case the Riley polynomial is $\phi_K(s, u) = u - (x^2 - 3)$, and hence

$$\tau_\rho(M) = -2\left(u^{-2}(u+1)(x^2-4) - 1\right)x^{-2} = \frac{2}{x^2(x^2-3)^2}.$$

(2) If $n = -1$, then $K = J(2, -2)$ is the figure eight knot. In this case the Riley polynomial is $\phi_K(s, u) = u^2 - (u+1)(x^2 - 5)$, and hence

$$\tau_\rho(M) = (2x-2)\left(u^{-2}(u+1)(x^2-4) - 1\right)x^{-2} = \frac{2x-2}{x^2(x^2-5)}.$$

The paper is organized as follows. In Section 2 we review the Chebyshev polynomials of the second kind and their properties. In Section 3 we give a formula for the Riley polynomial of a twist knot, and compute the trace of a canonical longitude. In Section 4 we review the Reidemeister torsion of a knot complement and its computation using Fox's free calculus. We prove Theorems 1 and 2 in Section 5.

2. Chebyshev polynomials

Recall that $S_k(z)$ are the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k. The following lemma is elementary.

Lemma 2.1. *One has $S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1$.*

Let $P_k(z) := \sum_{i=0}^k S_i(z)$.

Lemma 2.2. *One has $P_k(z) = \frac{S_{k+1}(z) - S_k(z) - 1}{z-2}$.*

Proof. We have

$$\begin{aligned} zP_k(z) &= z\sum_{i=0}^{k} S_i(z) = \sum_{i=0}^{k} \big(S_{i+1}(z)+S_{i-1}(z)\big) \\ &= \big(P_k(z)+S_{k+1}(z)-S_0(z)\big)+\big(P_k(z)-S_k(z)+S_{-1}(z)\big) \\ &= 2P_k(z)+S_{k+1}(z)-S_k(z)-1. \end{aligned}$$

The lemma follows. □

Lemma 2.3. *One has* $P_k^2(z)+P_{k-1}^2(z)-zP_k(z)P_{k-1}(z)=P_k(z)+P_{k-1}(z)$.

Proof. Let $Q_k(z)=\big(P_k^2(z)+P_{k-1}^2(z)-zP_k(z)P_{k-1}(z)\big)-\big(P_k(z)+P_{k-1}(z)\big)$. We have

$$Q_{k+1}(z)-Q_k(z)=\big(P_{k+1}(z)-P_{k-1}(z)\big)\big(P_{k+1}(z)+P_{k-1}(z)-zP_k(z)-1\big).$$

Since $zP_k(z)=\sum_{i=0}^{k}\big(S_{i+1}(z)+S_{i-1}(z)\big)=P_{k+1}(z)-1+P_{k-1}(z)$, we obtain $Q_{k+1}(z)=Q_k(z)$ for all integers k. Hence $Q_k(z)=Q_1(z)=0$. □

Proposition 2.4. *Suppose* $V=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\mathbb{C})$. *Then*

$$V^k = \begin{bmatrix} S_k(t)-dS_{k-1}(t) & bS_{k-1}(t) \\ cS_{k-1}(t) & S_k(t)-aS_{k-1}(t) \end{bmatrix}, \tag{2.1}$$

$$\sum_{i=0}^{k} V^i = \begin{bmatrix} P_k(t)-dP_{k-1}(t) & bP_{k-1}(t) \\ cP_{k-1}(t) & P_k(t)-aP_{k-1}(t) \end{bmatrix}, \tag{2.2}$$

where $t:=\operatorname{tr} V=a+d$. *Moreover, one has*

$$\det\left(\sum_{i=0}^{k} V^i\right) = \frac{S_{k+1}(z)-S_{k-1}(z)-2}{z-2}. \tag{2.3}$$

Proof. Since $\det V=1$, by the Cayley-Hamilton theorem we have $V^2-tV+I=0$. This implies that $V^k-tV^{k-1}+V^{k-2}=0$ for all integers k. Hence, by induction on k, one can show that $V^k=S_k(t)I-S_{k-1}(t)V^{-1}$. Since $V^{-1}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, (2.1) follows.

Since $P_k(t)=\sum_{i=0}^{k} S_i(t)$, (2.2) follows directly from (2.1). By Lemma 2.3 we have

$$\begin{aligned} \det\left(\sum_{i=0}^{k} V^i\right) &= P_k^2(t)+(ad-bc)P_{k-1}^2(t)-(a+d)P_k(t)P_{k-1}(t) \\ &= P_k^2(t)+P_{k-1}^2(t)-tP_k(t)P_{k-1}(t) \\ &= P_k(t)+P_{k-1}(t). \end{aligned}$$

Then (2.3) follows from Lemma 2.2. □

3. Nonabelian representations

In this section we give a formula for the Riley polynomial of a twist knot. This formula was already obtained in [DHY, Mo]. We also compute the trace of a canonical longitude.

3.1. **Riley polynomial.** Recall that $K = J(2,2n)$ and $E_K = S^3 \setminus K$. The fundamental group of E_K has a presentation $\pi_1(E_K) = \langle a, b \mid w^n a = b w^n \rangle$ where a, b are meridians and $w = ba^{-1}b^{-1}a$. Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ -u & s^{-1} \end{bmatrix}$$

where $(s,u) \in (\mathbb{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s,u)$.

We now compute $\phi_K(s,u)$. Since

$$\rho(w) = \begin{bmatrix} 1 - s^2 u & s^{-1} - s - su \\ (s - s^{-1})u + su^2 & 1 + (2 - s^{-2})u + u^2 \end{bmatrix},$$

by Lemma 2.4 we have

$$\rho(w^n) = \begin{bmatrix} S_n(z) - \big(1 + (2 - s^{-2})u + u^2\big) S_{n-1}(z) & (s^{-1} - s - su) S_{n-1}(z) \\ \big((s - s^{-1})u + su^2\big) S_{n-1}(z) & S_n(z) - (1 - s^2 u) S_{n-1}(z) \end{bmatrix},$$

where $z = \operatorname{tr} \rho(w) = 2 + (2 - s^2 - s^{-2})u + u^2$. Hence, by a direct computation we have

$$\rho(w^n a - b w^n) = \begin{bmatrix} 0 & \phi_K(s,u) \\ u\phi_K(s,u) & 0 \end{bmatrix}$$

where

$$\phi_K(s,u) = S_n(z) - \big(u^2 - (u+1)(s^2 + s^{-2} - 3)\big) S_{n-1}(z).$$

3.2. **Trace of the longitude.** It is known that the canonical longitude corresponding to the meridian $\mu = a$ is $\lambda = \overleftarrow{w}^n w^n$, where $\overleftarrow{w}$ is the word in the letters a, b obtained by writing w in the reversed order. We now compute its trace. This computation will be used in the proof of Theorem 2.

Lemma 3.1. *One has* $S_{n-1}^2(z) = \frac{1}{(u+2-s^2-s^{-2})(u^2-(s^2+s^{-2}-2)(u+1))}$.

Proof. Since $(s,u) \in (\mathbb{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s,u)$, we have $S_n(z) = (u^2 - (u+1)(s^2 + s^{-2} - 3)) S_{n-1}(z)$. Lemma 2.1 then implies that

$$\begin{aligned} 1 &= S_n^2(z) - z S_n(z) S_{n-1}(z) + S_{n-1}^2(z) \\ &= \Big(\big(u^2 - (u+1)(s^2 + s^{-2} - 3)\big)^2 - z\big(u^2 - (u+1)(s^2 + s^{-2} - 3)\big) + 1 \Big) S_{n-1}^2(z). \end{aligned}$$

By replacing $z = 2 + (2 - s^2 - s^{-2})u + u^2$ into the first factor of the above expression, we obtain the desired equality. $\square$

Proposition 3.2. *One has* $\operatorname{tr} \rho(\lambda) - 2 = \frac{u^2(s^2+s^{-2}+2)}{(u+1)(s^2+s^{-2}-2)-u^2}$.

Proof. By Lemma 2.4 we have

$$\rho(w^n) = \begin{bmatrix} S_n(z) - \big(1 + (2 - s^{-2})u + u^2\big) S_{n-1}(z) & (s^{-1} - s - su) S_{n-1}(z) \\ \big((s - s^{-1})u + su^2\big) S_{n-1}(z) & S_n(z) - (1 - s^2 u) S_{n-1}(z) \end{bmatrix}.$$

Similarly,

$$\rho(\overleftarrow{w}^n) = \begin{bmatrix} S_n(z) - (1 - s^{-2}u) S_{n-1}(z) & (s - s^{-1} - s^{-1}u) S_{n-1}(z) \\ \big((s^{-1} - s)u + s^{-1}u^2\big) S_{n-1}(z) & S_n(z) - \big(1 + (2 - s^2)u + u^2\big) S_{n-1}(z) \end{bmatrix}.$$

Hence, by a direct calculation we have

$$\begin{aligned}\operatorname{tr}\rho(\lambda) &= \operatorname{tr}(\rho(\overleftarrow{w}^{n})\rho(w)) \\ &= 2S_n^2(z) - 2zS_n(z)S_{n-1}(z) + \big(2 + (s^4 + s^{-4} - 2)u^2 - (s^2 + s^{-2} + 2)u^3\big)S_{n-1}^2(z) \\ &= 2 + u^2(s^2 + s^{-2} + 2)(s^2 + s^{-2} - 2 - u)S_{n-1}^2(z).\end{aligned}$$

The lemma then follows from Lemma 3.1. □

4. Reidemeister torsion

In this section we briefly review the Reidemeister torsion of a knot complement and its computation using Fox's free calculus. For more details on the Reidemeister torsion, see [Jo, Mi1, Mi2, Mi3, Tu].

4.1. **Torsion of a chain complex.** Let C be a chain complex of finite dimensional vector spaces over $\mathbb{C}$:

$$C = \Big(0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0\Big)$$

such that for each $i = 0, 1, \cdots, m$ the followings hold

- the homology group $H_i(C)$ is trivial, and
- a preferred basis c_i of C_i is given.

Let $B_i \subset C_i$ be the image of ∂_{i+1}. For each i choose a basis b_i of B_i. The short exact sequence of $\mathbb{C}$-vector spaces

$$0 \to B_i \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

implies that a new basis of C_i can be obtained by taking the union of the vectors of b_i and some lifts $\tilde{b}_{i-1}$ of the vectors b_{i-1}. Define $[(b_i \cup \tilde{b}_{i-1})/c_i]$ to be the determinant of the matrix expressing $(b_i \cup \tilde{b}_{i-1})$ in the basis c_i. Note that this scalar does not depend on the choice of the lift $\tilde{b}_{i-1}$ of b_{i-1}.

Definition 4.1. The *torsion* of C is defined to be

$$\tau(C) := \prod_{i=0}^{m} [(b_i \cup \tilde{b}_{i-1})/c_i]^{(-1)^{i+1}} \in \mathbb{C} \setminus \{0\}.$$

Remark 4.2. Once a preferred basis of C is given, $\tau(C)$ is independent of the choice of $b_0, \dots, b_m$.

4.2. **Reidemeister torsion of a CW-complex.** Let M be a finite CW-complex and $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ a representation. Denote by $\tilde{M}$ the universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C(\tilde{M}; \mathbb{Z})$ has the structure of a chain complex of left $\mathbb{Z}[\pi_1(M)]$-modules.

Let V be the 2-dimensional vector space $\mathbb{C}^2$ with the canonical basis $\{e_1, e_2\}$. Using the representation ρ, V has the structure of a right $\mathbb{Z}[\pi_1(M)]$-module which we denote by V_ρ. Define the chain complex $C(M; V_\rho)$ to be $C(\tilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} V_\rho$, and choose a preferred basis of $C(M; V_\rho)$ as follows. Let $\{u_1^i, \cdots, u_{m_i}^i\}$ be the set of i-cells of M, and choose a lift $\tilde{u}_j^i$ of each cell. Then $\{\tilde{u}_1^i \otimes e_1, \tilde{u}_1^i \otimes e_2, \cdots, \tilde{u}_{m_i}^i \otimes e_1, \tilde{u}_{m_i}^i \otimes e_2\}$ is chosen to be the preferred basis of $C_i(M; V_\rho)$.

A representation ρ is called *acyclic* if all the homology groups $H_i(M; V_\rho)$ are trivial.

Definition 4.3. The Reidemeister torsion $\tau_\rho(M)$ is defined as follows:

$$\tau_\rho(M) = \begin{cases} \tau(C(M;V_\rho)) & \text{if } \rho \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

4.3. **Reidemeister torsion of a knot complement and Fox's free calculus.** Let L be a knot in S^3 and E_L its complement. We choose a Wirtinger presentation for the fundamental group of E_L:

$$\pi_1(E_L) = \langle a_1, \dots, a_l \mid r_1, \dots, r_{l-1} \rangle.$$

Let $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$ be a representation. This map induces a ring homomorphism $\rho : \mathbb{Z}[\pi_1(E_L)] \to M_2(\mathbb{C})$, where $\mathbb{Z}[\pi_1(E_L)]$ is the group ring of $\pi_1(E_L)$ and $M_2(\mathbb{C})$ is the matrix algebra of degree 2 over $\mathbb{C}$. Consider the $(l-1)\times l$ matrix A whose (i,j)-component is the 2×2 matrix

$$\rho\left(\frac{\partial r_i}{\partial a_j}\right) \in M_2(\mathbb{C}),$$

where $\partial/\partial a$ denotes the Fox calculus. For $1 \le j \le l$, denote by A_j the $(l-1)\times(l-1)$ matrix obtained from A by removing the jth column. We regard A_j as a $2(l-1)\times 2(l-1)$ matrix with coefficients in $\mathbb{C}$. Then Johnson showed the following.

Theorem 4.4. [Jo] *Let $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$ be a representation such that* $\det(\rho(a_1) - I) \neq 0$. *Then the Reidemeister torsion of E_L is given by*

$$\tau_\rho(E_L) = \frac{\det A_1}{\det(\rho(a_1) - I)}.$$

5. Proof of main results

5.1. **Proof of Theorem 1.** We will apply Theorem 4.4 to calculate the Reidemeister torsion of the complement E_K of the twist knot $K = J(2,2n)$.

Recall that $\pi_1(E_K) = \langle a, b \mid w^n a = b w^n \rangle$. We have $\det(\rho(b) - I) = 2 - (s + s^{-1}) = 2 - x$. Let $r = w^n a w^{-n} b^{-1}$. By a direct computation we have

$$\begin{aligned} \frac{\partial r}{\partial a} &= w^n \left(1 + (1-a)(w^{-1} + \cdots + w^{-n}) \frac{\partial w}{\partial a}\right) \\ &= w^n \left(1 + (1-a)(1 + w^{-1} + \cdots + w^{-(n-1)}) a^{-1}(1-b)\right). \end{aligned}$$

Suppose $x \neq 2$. Then $\det(\rho(b) - I) \neq 0$ and hence

$$\tau_\rho(E_K) = \det \rho\left(\frac{\partial r}{\partial a}\right) \Big/ \det(\rho(b) - I) = \det \rho\left(\frac{\partial r}{\partial a}\right) \Big/ (2 - x).$$

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Let $\Delta = \rho(1 + w^{-1} + \cdots + w^{-(n-1)})$ and $\Omega = \rho\big(a^{-1}(1-b)(1-a)\big)\Delta$. Then

$$\det \rho\left(\frac{\partial r}{\partial a}\right) = \det(I + \Omega) = 1 + \det \Omega + \operatorname{tr} \Omega.$$

Lemma 5.1. *One has* $\det \Omega = (2-x)^2 \left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right)$.

Proof. Since $\operatorname{tr} \rho(w^{-1}) = \operatorname{tr} \rho(w) = z$, by Lemma 2.4 we have $\det \Delta = \frac{S_n(z) - S_{n-2}(z) - 2}{z-2}$. The lemma follows, since $\det \Omega = \det \rho(a^{-1}(1-a)(1-b)) \det \Delta = (2-x)^2 \det \Delta$. $\square$

Lemma 5.2. *One has* $\operatorname{tr}\Omega = x(2-x)S_{n-1}(z) - 1$.

Proof. Since $\rho(w^{-1}) = \begin{bmatrix} 1+(2-s^{-2})u+u^2 & s-s^{-1}+su \\ (s^{-1}-s)u-su^2 & 1-s^2u \end{bmatrix}$, by Lemma 2.4 we have

$$\Delta = \begin{bmatrix} P_{n-1}(z)-(1-s^2u)P_{n-2}(z) & (s-s^{-1}+su)P_{n-2}(z) \\ \big((s^{-1}-s)u-su^2\big)P_{n-2}(z) & P_{n-1}(z)-\big(1+(2-s^{-2})u+u^2\big)P_{n-2}(z) \end{bmatrix}.$$

By a direct computation we have

$$\rho(a^{-1}(1-b)(1-a)) = \begin{bmatrix} s+s^{-1}-2+(s-1)u & s^{-1}-s^{-2}+u \\ su-s^2u & s+s^{-1}-2-su \end{bmatrix}.$$

Hence

$$\begin{aligned}
\operatorname{tr}\Omega &= \operatorname{tr}\big(\rho\big(a^{-1}(1-b)(1-a)\big)\Delta\big) \\
&= (2s+2s^{-1}-4-u)P_{n-1}(z)+\big(4-2s-2s^{-1}+(3-s^2-s^{-2})u+u^2\big)P_{n-2}(z) \\
&= (2s+2s^{-1}-4-u)\big(P_{n-1}(z)-P_{n-2}(z)\big)+((2-s^2-s^{-2})u+u^2)P_{n-2}(z) \\
&= (2s+2s^{-1}-4-u)S_{n-1}(z)+(z-2)P_{n-2}(z) \\
&= (2s+2s^{-1}-4-u)S_{n-1}(z)+S_{n-1}(z)-S_{n-2}(z)-1.
\end{aligned}$$

Since (s,u) satisfies $\phi_K(s,u)=0$, we have $S_n(z) = \left(u^2-(u+1)(s^2+s^{-2}-3)\right)S_{n-1}(z)$. This implies that $S_{n-2}(z) = zS_n(z)-S_{n-1}(z) = (s^2+s^{-2}-1-u)S_{n-1}(z)$. Hence

$$\operatorname{tr}\Omega = \big(2s+2s^{-1}-s^2-s^{-2}-2\big)S_{n-1}(z).$$

The lemma follows since $2s+2s^{-1}-s^2-s^{-2}-2 = x(2-x)$. □

We now complete the proof of Theorem 1. Lemmas 5.1 and 5.2 imply that

$$\det\rho\left(\frac{\partial r}{\partial a}\right) = 1+\det\Omega+\operatorname{tr}\Omega = (2-x)^2\left(\frac{S_n(z)-S_{n-2}(z)-2}{z-2}\right)+x(2-x)S_{n-1}(z).$$

Since $\tau_\rho(E_K) = \det\rho\left(\frac{\partial r}{\partial a}\right)/(2-x)$, we obtain the desired formula for $\tau_\rho(E_K)$.

Remark 5.3. In [Mo], Morifuji proved a similar formula for the twisted Alexander polynomial of twist knots for nonabelian representations.

5.2. **Proof of Theorem 2.** Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbb{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. Recall that λ is the canonical longitude corresponding to the meridian $\mu = a$. If $\operatorname{tr}\rho(\lambda) \neq 2$, then by [Ki1] (see also [Ki2, Ki3]) the Reidemeister torsion of M is given by

$$\tau_\rho(M) = \frac{\tau_\rho(E_K)}{2-\operatorname{tr}\rho(\lambda)}. \tag{5.1}$$

By Theorem 1 we have $\tau_\rho(E_K) = (2-x)\frac{S_n(z)-S_{n-2}(z)-2}{z-2}+xS_{n-1}(z)$ if $x \neq 2$. By Proposition 3.2 we have $\operatorname{tr}\rho(\lambda)-2 = \frac{x^2}{u^{-2}(u+1)(x^2-4)-1}$. Theorem 2 then follows from (5.1).

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